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Time-Series Analysis of a Collective Variable in High-Dimensional Cellular Automata

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We perform a maximum-entropy frequency analysis of the occupation-density time series for a recently proposed totalistic cellular automaton rule in five dimensions. This new information complements partial knowledge coming from winding number measurements. We discuss the possible phenomenology of the model in terms of our findings.

KEY WORDS: Time-series analysis; deterministic cellular automata.

The emergence of collective behavior in high-dimensional cellular automata (CA) has been established recently by Chaté and Manneville[1,2] and confirmed by others.[3-5] This result apparently challenges droplet-type arguments by Bennett *et al.*[6] which limit drastically the possible patterns of stable collective states in dimensions $d \geqslant 2$ to be at most period-two. In particular, refs. 1–5 report simulations of many high-dimensional CA models in which the average (collective) variable settles to a period-three (or four) cycle, and more mysteriously, quasiperiodic behavior. An additional interesting feature of the models is the competition between the mean-field tendencies in high dimensions and the correlation buildup one expects in deterministic systems.

In this short communication we analyze the frequency spectrum of the time series corresponding to the collective variable of one of the models reported in the literature. We find that this variable is well-described by a Fourier-type series with a fundamental frequency which apparently is *not* simply related to the intrinsic time scale of the system. Our results are consistent with winding-number measurements of the times series, and clarify and extend what has been reported in the literature. For example, we determine how many terms in the series are necessary for a satisfactory reconstruction

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of the return map. We close the paper with a discussion of the possible phenomenology of the system, again clarifying what has been discussed by other authors.

The model we consider was originally proposed in ref. 1. It is a five-dimensional, deterministic two-state (zero or one) cellular automaton. At each time step a local field h_i is computed for each site, over the site itself and its ten nearest neighbors; this corresponds to the five-dimensional von Neumann zone. The sites are updated *synchronously*, and given the value one if their local field is $5 \leqslant h_i \leqslant 8$, and zero otherwise. As it has been established that the system is robust to the choice of boundary conditions, we have chosen periodic for simplicity. Updating the system at integer time steps defines an intrinsic time scale $t = 1$.

A time series for the collective variable

$$S(t) = \frac{1}{L^5} \sum_{\text{sites}} s(t)$$

has been recorded for $t > 5000$, and the corresponding frequency spectrum has been found for several system sizes L. The most probable spectrum was determined using the Bayesian maximum entropy method, in which, under very general conditions, a spectrum's prior probability is assigned to be proportional to the exponential of its configurational entropy. Bayes' theorem is used to incorporate the data and provide a posterior probability distribution of feasible spectra. The displayed spectrum is the one which maximizes this probability. For convenience, the Hartley spectrum[7] was the quantity that was determined. From it, it is easy to reconstruct a Fourier amplitude spectrum, which is what is shown here. (See ref. 8 for further discussion of the details of this method.)

The main result is given in Fig. 1, which shows the frequency spectrum for 1000 time steps of $S(t)$ and a lattice of 10^5 sites. The spectrum consists of a large fundamental mode at $f \sim 0.3476 \pm (2)$, with second and third harmonics showing at $f \sim 0.3$, 0.04. These and higher harmonics appear reflected about the Nyquist critical frequency $f = 1/2$. The frequency spectrum for other system sizes is essentially identical. Our best estimate for the fundamental frequency, obtained from a 10,000 point time series, is $f \sim 0.34805 \pm (5)$, which has been confirmed by winding number measurements.[9] Information about a fit of the noise-reduced time series to the form

$$S(t) = \sum C_n \operatorname{cas}(nft + \phi_n) \tag{1}$$

is given in Table I. Our results, showing that the time series is well described by a periodic signal of frequency apparently irrationally related to the iteration time, are consistent with the iterated-map use of the term "quasiperiodic."

We have been successful in reconstructing the skeleton of the model's return map ($S(t+1)$ vs. $S(t)$). This requires three or more of the terms in the Fourier-type expansion, as one term yields an ellipse and two a

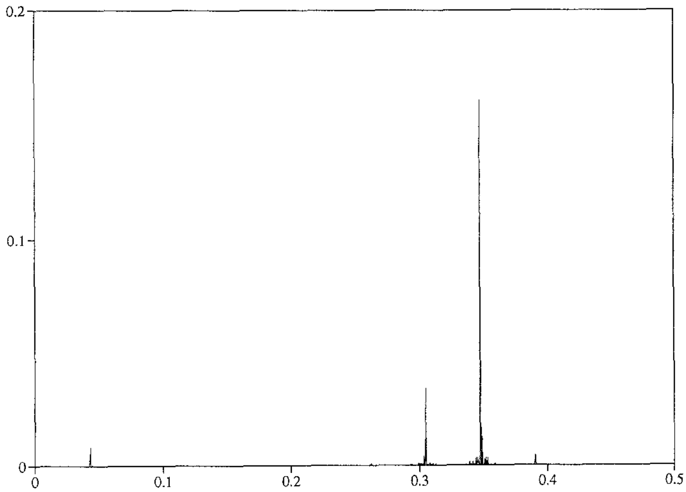

Fig. 1. Frequency spectrum of the 5-dimensional cellular automaton, showing the fundamental frequency $f \sim 0.348$ and its harmonics. Amplitude and phase data are also given in Table I.

Lissajous-like curve. A six-term fit essentially reproduces the original return map, minus the thermodynamic-like noise.

We close this short communication with two proposals on how to approach analytically the phenomenon of collective behavior in this model. One is that the time series could be a transient, closely following (shadowing) a limit cycle of integer period. If this is so, Jen's studies[10] of shifting mechanisms in limit cycles for one-dimensional CA could be relevant. However, analogous studies in high dimensions have never been undertaken. The corresponding limit cycle, if it exists, might be quite long. In fact, if we estimate (see, for example, ref. 11 and references therein) the length of a typical limit cycle or transient to be of the order of the square root of the number of microstates, a typical limit cycle for this system might be of length 2^{50000}.

It is, however, more likely that the fundamental frequency is irrationally related to the intrinsic time scale of the system. This is supported by the

Table I. Frequency (divided by $f_o \sim 0.348$), amplitude C_n and phase ϕ_n of the first six terms in the series expansion of the time series for the 5*d* cellular automaton

$f/(f_o)$	C_n	ϕ_n
1	$5.1288e-2$	-2.2382
2	$1.4398e-2$	1.8585
3	$5.1907e-3$	-0.1057
4	$3.3630e-3$	1.3908
5	$1.8045e-3$	-1.4838
6	$7.6711e-4$	2.2257

fact that this frequency appears to be independent of system size. In this case, a continuous-time description of the evolution of the collective modes might be necessary. It is not clear how this description would be implemented for a synchronously updated system, but the simplicity of the behavior of the collective variable suggests that the mechanism producing the time series can be elucidated. In addition, the discrete second-order time dynamics of a single order parameter can yield an irrational period.[12] This is an appealing alternative to the two order parameters proposed in refs. 1 and 2, which seems more appropriate to describe continuous quasiperiodic behavior with two independent frequencies rather than one (see ref. 13, chapter III). In this respect, it is well known that a one-variable iteration such as the circle map[3,14] is sufficient to produce quasiperiodic behavior in the sense used in the iterated-map literature.

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NOTE ADDED IN PROOF

Bennett *et al.* (Ref. 6) specifically exclude irrational periods from their proof. In equation (1) of the present paper, $\text{cas}\ x = \cos x + \sin x$.

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